Canonical q-deformations of noncompact Lie (super-)algebras



1993 J. Phys. A: Math. Gen. 26 1317

(http://iopscience.iop.org/0305-4470/26/6/015)

J. Phys. A: Math. Gen. 26 (1993) 1317–1334. Printed in the UK

# Canonical $q$-deformations of non-compact Lie (super-)algebras

V K Dobrev†

Göttingen University, Institute for Theoretical Physics, Bunsenstrasse 9, 3400 Göttingen, Federal of Germany
and
International Centre for Theoretical Physics, PO Box 586, 34100 Trieste, Italy

Received 24 February 1992, in final form 3 August 1992

**Abstract.** We propose a procedure for $q$-deformations of the real forms $\mathcal{G}$ of complex Lie (super-)algebras associated with (generalized) Cartan matrices. Our procedure gives different $q$-deformations for the non-conjugate Cartan subalgebras of $\mathcal{G}$. We give several illustrations, e.g., $q$-deformed Lorentz and conformal (super-)algebras. The $q$-deformed conformal algebra contains as a subalgebra a $q$-deformed Poincaré algebra and as Hopf subalgebras two conjugate 11-generator $q$-deformed Weyl algebras. The $q$-deformed Lorentz algebra is a Hopf subalgebra of both Weyl algebras.

## 1. Introduction

Non-compact Lie groups and algebras play a very important role in physics—recall, e.g., the Lorentz, Poincaré, conformal groups. Thus ever since the introduction of quantum groups as deformations $U_q(\mathcal{G})$ of the universal enveloping algebras of complex simple Lie algebras [1, 2], or as matrix quantum groups [3–5], one was always asking what would be the deformation of the real forms. In fact, the deformation of *compact* simple Lie algebras is used in the physics literature without much explanation assuming implementation of the Weyl unitary trick. In [3] the compact matrix quantum groups $SU_q(n)$, (for $n = 2$ first in [5]), $SO_q(n)$, $Sp_q(n)$ and the maximally split real non-compact forms $SL_q(n,\mathbb{R})$, $SO_q(n,n)$, $SO_q(n,n+1)$, $Sp_q(n,\mathbb{R})$ were introduced. From our point of view it is not accidental that these cases were obtained first since the root systems of these real forms coincide (up to multiple of $i$ in the compact case) with the root systems of their complexifications (see the description of our approach below). Besides the above $U_q(su(1,1))$ was considered in [6], $U_q(su(n,1))$ were introduced in [7]. A quantum Lorentz group was introduced and studied in [8] and a seven-dimensional quantum Lorentz algebra was introduced in [9].

Thus there is still no universal approach to the $q$-deformation of real simple algebras. Such an approach is proposed in the present paper. Let $\mathcal{G}$ be a real simple Lie algebra (we shall need to extend the construction to real reductive Lie algebras later). We shall use the standard deformation (cf section 2.2) for the simple

† A von Humboldt Foundation fellow; permanent address: Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria.

0305-4470/93/061317+18$07.50  

components of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$ to obtain deformations $U_q(\mathcal{G})$ as a real form of $U_q(\mathcal{G}^{\mathbb{C}})$. Though the procedure is described mostly in terms which are known from the undeformed case we stress which steps are necessitated by the $q$-deformation. The first basic ingredient of our approach relies on the fact that the real forms $\mathcal{G}$ of a complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$ are in 1-to-1 correspondence with the Cartan automorphisms $\theta$ of $\mathcal{G}^{\mathbb{C}}$. This allows us to study the structure of the real forms and to find their explicit embeddings as real subalgebras of $\mathcal{G}^{\mathbb{C}}$ invariant under $\theta$ and, consequently, using the same generators, to find $U_q(\mathcal{G})$. This ingredient is enough for the compact case (up to the choice of range of $q$). The second basic ingredient is related to the fact that a real non-compact simple Lie algebra has in general (a finite number of) non-conjugate Cartan subalgebras [10]. This is very important since we have to choose which conjugacy class of Cartan subalgebras will correspond to the unique conjugacy class of Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$ and will be 'frozen' under a $q$-deformation (see (4*a*)). For each such choice we shall get a different $q$-deformation. The third basic ingredient are the Bruhat decompositions $\mathcal{G} = \mathcal{A} \oplus \mathcal{M} \oplus \tilde{\mathcal{N}} \oplus \mathcal{N}$ (direct sum of vector subspaces), where $\mathcal{A}$ is a non-compact Abelian subalgebra, $\mathcal{M}$ (a reductive Lie algebra) is the centralizer of $\mathcal{A}$ in $\mathcal{G}$ (mod $\mathcal{A}$), and $\tilde{\mathcal{N}}$, $\mathcal{N}$, respectively, are nilpotent subalgebras forming the positive, negative, respectively, root spaces of the root system $(\mathcal{G}, \mathcal{A})$. Consistently, the Cartan subalgebras of $\mathcal{G}$ have the decomposition $\mathcal{H} = \mathcal{A} \oplus \mathcal{H}^m$, where $\mathcal{H}^m$ is a Cartan subalgebra of $\mathcal{M}$. A general property of the deformations $U_q(\mathcal{G})$ obtained by our procedure is that $U_q(\mathcal{M})$, $U_q(\tilde{\mathcal{P}})$, $U_q(\mathcal{P})$ are Hopf subalgebras of $U_q(\mathcal{G})$, where $\mathcal{P} = \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N}$, $\tilde{\mathcal{P}} = \mathcal{A} \oplus \mathcal{M} \oplus \tilde{\mathcal{N}}$, are parabolic subalgebras of $\mathcal{G}$. Our approach is easily generalized for the real forms of the basic classical Lie superalgebras and of the corresponding affine Kac–Moody (super-)algebras.

These $q$-deformations are called canonical because they are obtained by a well defined procedure presented below. This does not exclude other deformations, e.g., multiparameter deformations or deformation by contraction (cf also comments in the text). Also, as in the undeformed case for each real form, there exists an antilinear (anti)involution $\sigma$ of $U_q(\mathcal{G}^{\mathbb{C}})$ which preserves $U_q(\mathcal{G})$. Unlike the undeformed case it is necessary to consider both involutions and anti-involutions, since there are two possibilities for the deformation parameter $q$, i.e. either $|q| = 1$ or $q \in \mathbb{R}$. For instance, $U_q(\mathrm{su}(2))$ has $|q| = 1$ when $\sigma$ is an involution and $q \in \mathbb{R}$ when $\sigma$ is an anti-involution. Further, $\sigma$ is a coalgebra (anti)homomorphism, i.e. $\delta \circ \sigma = (\sigma \times \sigma) \circ \delta$, or $\delta \circ \sigma = (\sigma \times \sigma) \circ \delta'$; $\varepsilon(\sigma(X)) = \bar{\varepsilon}(X)\ \forall X \in U_q(\mathcal{G}^{\mathbb{C}})$. Then the relations for the antipode are: $\sigma \circ \gamma = \gamma \circ \sigma$ if $\sigma$ is an algebra involution and a coalgebra homomorphism or if it is an algebra anti-involution and a coalgebra antihomomorphism and $(\sigma \circ \gamma)^2 = \mathrm{id}$ otherwise. One approach to the real forms would be to try to classify the possible conjugations $\sigma$ directly. Our approach is more constructive and the conjugations $\sigma$ are obtained as a byproduct of the procedure proposed below (this is pointed out in some examples).

The organization of the paper is as follows. In section 2 we recall some basic facts about real simple Lie algebras and about the $q$-deformation of complex simple Lie algebras. In section 3 we present our approach. In section 4 we present a $q$-deformation of the algebras $\mathrm{so}(p, r)$, in particular, the Lorentz algebra $\mathrm{so}(3, 1)$. In section 5 we present a $q$-deformation of the conformal algebra $\mathrm{su}(2, 2)$ containing—cf section 6—as a subalgebra a $q$-deformed Poincaré algebra and as Hopf subalgebras two conjugate 11-generator $q$-deformed Weyl algebras. In section 7 we recall the

$q$-deformation of complex Lie superalgebras and present a $q$-deformation of the conformal superalgebra su$(2,2/N)$.

## 2. Preliminaries

### *2.1. Real semi-simple Lie algebras [10]*

Let $\mathcal{G}$ be a real semi-simple Lie algebra, $\theta$ be the Cartan involution in $\mathcal{G}$, and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the Cartan decomposition of $\mathcal{G}$, so that $\theta X = X, X \in \mathcal{K}$, $\theta X = -X, X \in \mathcal{P}$; $\mathcal{K}$ is the maximal compact subalgebra of $\mathcal{G}$. Let $\mathcal{A}_0$ be the maximal subspace of $\mathcal{P}$ which is an Abelian subalgebra of $\mathcal{G}$; $r_0$ =dim $\mathcal{A}_0$ is the *real* (or *split*) rank of $\mathcal{G}$, $0 \leqslant r_0 \leqslant \ell =$ rank $\mathcal{G}$.

Let $\Delta_R^0$ be the root system of the pair $(\mathcal{G}, \mathcal{A}_0)$, also called ($\mathcal{A}_0$-) *restricted root system*:

$$\Delta_R^0 = \{\lambda \in \mathcal{A}_0^* | \lambda \neq 0, \mathcal{G}_\lambda^0 \neq 0\} \qquad \mathcal{G}_\lambda^0 = \{X \in \mathcal{G} | [Y, X] = (Y)X, \forall Y \in \mathcal{A}_0\}. \quad (1)$$

The elements of $\Delta_R^0 = \Delta_R^{0+} \cup \Delta_R^{0-}$ are called ($\mathcal{A}_0$-) *restricted roots*; if $\lambda \in \Delta_R^0$, $\mathcal{G}_\lambda^0$ are called ($\mathcal{A}_0$-) *restricted root spaces*, $\dim_R \mathcal{G}_\lambda^0 \geqslant 1$. Now we can introduce the subalgebras corresponding to the positive ($\Delta_R^{0+}$) and negative ($\Delta_R^{0-}$) restricted roots:

$$\tilde{\mathcal{N}}_0 = \bigoplus_{\lambda \in \Delta_R^{0+}} \mathcal{G}_\lambda^0 = \tilde{\mathcal{N}}_0^1 \oplus \tilde{\mathcal{N}}_0^2 \qquad \mathcal{N}_0 = \bigoplus_{\lambda \in \Delta_R^{0-}} \mathcal{G}_\lambda^0 = \mathcal{N}_0^1 \oplus \mathcal{N}_0^2 = \theta \tilde{\mathcal{N}}_0 \quad (2)$$

where $\tilde{\mathcal{N}}_0^1$, $\tilde{\mathcal{N}}_0^2$, respectively, are the direct sum of $\mathcal{G}_\lambda^0$ with $\dim_R \mathcal{G}_\lambda^0 = 1$, $\dim_R \mathcal{G}_\lambda^0 > 1$, respectively, and analogously for $\mathcal{N}_0^a = \theta \tilde{\mathcal{N}}_0^a$. Then we have the (Bruhat) decompositions which we shall use for our $q$-deformations:

$$\mathcal{G} = \tilde{\mathcal{N}}_0 \oplus \mathcal{A}_0 \oplus \mathcal{M}_0 \oplus \mathcal{N}_0 = \tilde{\mathcal{N}}_0^1 \oplus \tilde{\mathcal{N}}_0^2 \oplus \mathcal{A}_0 \oplus \mathcal{M}_0 \oplus \mathcal{N}_0^1 \oplus \mathcal{N}_0^2 \quad (3)$$

where $\mathcal{M}_0$ is the centralizer of $\mathcal{A}_0$ in $\mathcal{K}$, i.e. $\mathcal{M}_0 = \{X \in \mathcal{K} | [X, Y] = 0, \forall Y \in \mathcal{A}_0\}$. In general $\mathcal{M}_0$ is a compact reductive Lie algebra, and we shall write $\mathcal{M}_0 = \mathcal{M}_0^s \oplus \mathcal{Z}_0^m$, where $\mathcal{M}_0^s = [\mathcal{M}_0, \mathcal{M}_0]$ is the semi-simple part of $\mathcal{M}_0$, and $\mathcal{Z}_0^m$ is the centre of $\mathcal{M}_0$. Note that $\tilde{\mathcal{P}}_0^0 \equiv \tilde{\mathcal{N}}_0 \oplus \mathcal{A}_0 \oplus \mathcal{M}_0$, $\mathcal{P}_0^0 \equiv \mathcal{A}_0 \oplus \mathcal{M}_0 \oplus \mathcal{N}_0$ are subalgebras of $\mathcal{G}$, the so called minimal parabolic subalgebras of $\mathcal{G}$ for that choice of Cartan subalgebra.

All notions above are easily generalized for the real forms of the basic classical Lie superalgebras [11].

### *2.2. q-deformation in the complex case*

Let $\mathcal{G}_c$ be a complex simple Lie algebra; then the $q$-deformation $\mathrm{U}_q(\mathcal{G}_c)$ of the universal enveloping algebras $\mathrm{U}(\mathcal{G}_c)$ is defined [1,2] as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^\pm, H_j, j = 1, \ldots, \ell =$ rank $\mathcal{G}_c$ and with relations:

$$[H_j, H_k] = 0 \qquad [H_j, X_k^\pm] = \pm a_{jk} X_k^\pm$$
$$[X_j^+, X_k^-] = \delta_{jk} \frac{q_j^{H_j/2} - q_j^{-H_j/2}}{q_j^{1/2} - q_j^{-1/2}} = \delta_{jk} [H_j]_{q_j} \quad (4a)$$

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m}_{q_j} (X_j^\pm)^m X_k^\pm (X_j^\pm)^{n-m} = 0$$
$$j \neq k \qquad n = 1 - a_{jk} \qquad q_j = q^{(\alpha_j, \alpha_j)/2} \quad (4b)$$

where $(a_{jk}) = (2(\alpha_j, \alpha_k)/(\alpha_j, \alpha_j))$ is the Cartan matrix of $\mathcal{G}_c$, the scalar product of the roots $(\cdot, \cdot)$ is normalized so that

$$(\alpha, \alpha) \in 2\mathbb{N} \qquad \binom{n}{m}_q = [n]_q!/[m]_q![n-m]_q! \qquad [m]_q! = [m]_q[m-1]_q \ldots [1]_q$$

$$[m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} = \frac{\sinh(mh/2)}{\sinh(h/2)} = \frac{\sin(\pi m \tau)}{\sin(\pi\tau)}$$
$$q = \mathrm{e}^h = \mathrm{e}^{2\pi i \tau} \qquad h, \tau \in \mathbb{C}. \tag{5}$$

The elements $H_j$ span the Cartan subalgebra $\mathcal{H}_c$ of $\mathcal{G}_c$, while the elements $X_j^\pm$ generate the subalgebras $\mathcal{G}_c^\pm = \underset{\beta\in\Delta^\pm}{\oplus} \mathcal{G}_{c\beta}$, where $\Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}_c$, $\Delta_S$ will denote the set of simple roots of $\Delta$. One has the standard decomposition $\mathcal{G}_c = \mathcal{G}_c^+ \oplus \mathcal{H}_c \oplus \mathcal{G}_c^-$. We recall that the $H_j$ correspond to the simple roots $\alpha_j$ of $\mathcal{G}_c$, and if $\beta^\vee = \sum_j n_j \alpha_j^\vee$, $\beta^\vee \equiv 2\beta/(\beta,\beta)$, then to $\beta$ corresponds $H_\beta = \sum_j n_j H_j$. The elements of $\mathcal{G}_c$ which span $\mathcal{G}_{c\beta}$, (dim $\mathcal{G}_{c\beta} = 1$), are denoted by $X_\beta$. These Cartan–Weyl generators $H_\beta$, $X_\beta$ [2, 12] are normalized so that

$$[X_\beta, X_{-\beta}] = [H_\beta]_{q_\beta} \qquad [H_\beta, X_{\pm\beta'}] = \pm(\beta^\vee, \beta')X_{\pm\beta'} \qquad \beta, \beta' \in \Delta^+$$
$$q_\beta \equiv q^{(\beta,\beta)/2}. \tag{6}$$

The algebra $\mathrm{U}_q(\mathcal{G}_c)$ is a Hopf algebra [13] with co-multiplication $\delta$, co-unit $\varepsilon$ (homomorphisms) and antipode $\gamma$ (antihomomorphism) defined on the generators of $\mathrm{U}_q(\mathcal{G}_c)$ as follows [1, 2]

$$\delta(H_j) = H_j \otimes 1 + 1 \otimes H_j \qquad \delta(X_j^\pm) = X_j^\pm \otimes q_j^{H_j/4} + q_j^{-H_j/4} \otimes X_j^\pm \tag{7a}$$

$$\varepsilon(H_j) = \varepsilon(X_j^\pm) = 0 \qquad \gamma(H_j) = -H_j$$
$$\gamma(X_j^\pm) = -q_j^{\pm\hat\rho/2} X_j^\pm q_j^{\mp\hat\rho/2} = -q_j^{\pm 1/2} X_j^\pm \tag{7b}$$

where $\hat\rho \in \mathcal{H}_c$ corresponds to $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$, $\hat\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} H_\alpha$. The action of $\delta$, $\varepsilon$, $\gamma$ on the Cartan–Weyl generators $H_\beta$, $X_\beta$ is obtained easily from (6) since $H_\beta$ (see above) and $X_\beta$ (cf [2, 12] and, e.g. formulae (22), (25)) are given algebraically in terms of the Chevalley generators. (Note that if $\alpha \notin \Delta_S$ the coalgebra operations $\delta$, $\gamma$ look more complicated than (7).)

## 3. $q$-deformation of the real forms

The exposition of our procedure is organized as follows. We fix a real simple Lie algebra $\mathcal{G}$ and its most non-compact Cartan subalgebra $\mathcal{H}_0$; then we present the procedure since it is most simple in this case (section 3.1). Then we point out the modifications necessary in order to consider Cartan subalgebras $\mathcal{H}$ of $\mathcal{G}$ which are non-conjugate to $\mathcal{H}_0$ (section 3.2). Until this moment we consider only the so called minimal parabolic subalgebras $\mathcal{P}_0$ (which are different for non-conjugate Cartan subalgebras). Next, for an arbitrary Cartan subalgebra, we extend the procedure for arbitrary parabolic subalgebras. Finally we note that we need to generalize the whole procedure to reductive Lie algebras which is straightforward (section 3.3).

### *3.1. Real form with most non-compact Cartan subalgebra*

Let $\mathcal{G}$ be a real semi-simple Lie algebra, and let us use the data given in section 2.1. The first step in our procedure is the choice of Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$.

Let $\mathcal{H}_0^m$ be the Cartan subalgebra of $\mathcal{M}_0$, i.e. $\mathcal{H}_0^m = \mathcal{H}_0^{ms} \oplus \mathcal{Z}_0^m$, where $\mathcal{H}_0^{ms}$ is the Cartan subalgebra of $\mathcal{M}_0^s$. Then $\mathcal{H}_0 \equiv \mathcal{H}_0^m \oplus \mathcal{A}_0$ is a Cartan subalgebra of $\mathcal{G}$, the most non-compact one; $\dim_R \mathcal{H}_0 = \dim_R \mathcal{H}_0^{ms} + \dim_R \mathcal{Z}_0^m + r_0$. We also choose $\mathcal{H}_0$ to be the Cartan subalgebra of $U_q(\mathcal{G})$. Let $\mathcal{H}^{\mathbb{C}}$ be the complexification of $\mathcal{H}_0$, ($\ell =$ rank $\mathcal{G}^{\mathbb{C}} = \dim_C \mathcal{H}^{\mathbb{C}}$); then it is a Cartan subalgebra of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$.

The second step in our procedure is to choose consistently the basis of the rest of $\mathcal{G}$ and $\mathcal{G}^{\mathbb{C}}$, and thus of $U_q(\mathcal{G})$. For this we use the classification of the roots from $\Delta$ with respect to $\mathcal{H}_0$. The set $\Delta_r^0 \equiv \{\alpha \in \Delta | \alpha|_{\mathcal{H}_0^m} = 0\}$ is called the set of *real roots*, $\Delta_i^0 \equiv \{\alpha \in \Delta | \alpha|_{\mathcal{A}_0} = 0\}$—the set of *imaginary roots*, $\Delta_c^0 \equiv \Delta \backslash (\Delta_r^0 \cup \Delta_i^0)$—the set of *complex roots* [10]. Thus $\Delta = \Delta_r^0 \cup \Delta_i^0 \cup \Delta_c^0$. Further, let $\alpha \in \Delta^+$, let $\mathcal{L}_\alpha^c$ be the complex linear span of $H_\alpha, X_\alpha, X_{-\alpha}$, and let $\mathcal{L}_\alpha = \mathcal{L}_\alpha^c \cap \mathcal{G}$. Then $\dim_R \mathcal{L}_\alpha = 3$ iff the $\alpha \in \Delta_r^0 \cup \Delta_i^0$ [10]. If $\alpha \in \Delta_r^0$ then $X_\alpha \in \mathcal{P}^{\mathbb{C}}$ and $\mathcal{L}_\alpha$ is non-compact. Since the Cartan subalgebra is $\mathcal{H}_0$, then $X_\alpha \in \mathcal{K}^{\mathbb{C}}$ and $\mathcal{L}_\alpha$ is compact if $\alpha \in \Delta_i^0 = \Delta_k^0$. The algebras $\mathcal{L}_\alpha$ are given by

$$\mathcal{L}_\alpha = \text{RLS}\{H_\alpha, X_\alpha, X_{-\alpha}\} \qquad \alpha \in \Delta_r^{0+} \tag{8a}$$

$$\mathcal{L}_\alpha = \text{RLS}\{\mathrm{i}H_\alpha, X_\alpha - X_{-\alpha}, \mathrm{i}(X_\alpha + X_{-\alpha})\} \qquad \alpha \in \Delta_k^{0+} \tag{8b}$$

where RLS stands for real linear span.

Note that there is a 1-to-1 correspondence between the real roots $\alpha \in \Delta_r^0$ and the restricted roots $\lambda \in \Delta_R^0$ with $\dim_R \mathcal{G}_\lambda^0 = 1$ and naturally this correspondence is realized by the restriction: $\lambda = \alpha|_{\mathcal{A}_0}$. Thus the elements in (8*a*) $X_\alpha^\pm$ for $\alpha \in \Delta_r^0$ we take also as elements of $U_q(\mathcal{G})$. These generators obey

$$[X_\alpha, X_{-\alpha}] = [H_\alpha]_{q_\alpha} \qquad [H_\alpha, X_{\pm\alpha}] = \pm\alpha(H_\alpha)X_{\pm\alpha} \qquad \text{for } \alpha \in \Delta_r^{0+} \tag{9}$$

and the Hopf algebra structure is given exactly as for $\alpha \in \Delta$, cf (7) and the text after that.

*Remark 1. Formulae (8a), (9) determine completely a q-deformation of any maximally split real form (or normal real form),* when all roots are real, $\mathcal{M}_0 = 0$, and $\mathcal{H}_0 = \mathcal{A}_0$. In this case the decompostion (3) is just

$$\mathcal{G} = \tilde{\mathcal{N}}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \tag{10}$$

i.e. this is the restriction to $\mathbb{R}$ of the standard decomposition $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+^{\mathbb{C}} \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}_-^{\mathbb{C}}$, and hence $U_q(\mathcal{G})$ is just the restriction of $U_q(\mathcal{G}^{\mathbb{C}})$ to $\mathbb{R}$ with $q \in \mathbb{R}$. Thus we also inherit the property that $U_q(\tilde{\mathcal{N}}_0 \oplus \mathcal{A}_0)$, $U_q(\mathcal{N}_0 \oplus \mathcal{A}_0)$ are Hopf subalgebras of $U_q(\mathcal{G})$, since $U_q(\mathcal{G}_\pm^{\mathbb{C}} \oplus \mathcal{H}^{\mathbb{C}})$ are Hopf subalgebras of $U_q(\mathcal{G}^{\mathbb{C}})$. Note that $\sigma$ here is an antilinear involution and coalgebra homomorphism such that $\sigma(Y) = Y$ $\forall Y \in U_q(\mathcal{G}^{\mathbb{C}})$. For the classical complex Lie algebras these forms are $U_q(\mathrm{sl}(n,\mathbb{R}))$, $U_q(\mathrm{so}(n,n))$, $U_q(\mathrm{so}(n+1,n))$, $U_q(\mathrm{sp}(n,\mathbb{R}))$, which are dual to the matrix quantum groups $SL_q(n,\mathbb{R})$, $SO_q(n,n)$, $SO_q(n,n+1)$, $Sp_q(n,\mathbb{R})$, introduced in [3] from a different point of view from ours.

Further note that the set of the imaginary roots $\Delta_i^0$ may be identified with the root system of $\mathcal{M}_0^{s\mathbb{C}}$. Thus the elements in (8*b*) give the Hopf algebra $\mathrm{U}_q(\mathcal{M}_0^s)$ by the formulae

$$[C_\alpha^+, C_\alpha^-] = \frac{\sinh(\tilde{H}_\alpha h_\alpha/2)}{\sin(h_\alpha/2)} \qquad [\tilde{H}_\alpha, C_\alpha^\pm] = \pm C_\alpha^\mp \qquad q_\alpha = q^{(\alpha,\alpha)/2} = \mathrm{e}^{-\mathrm{i}h_\alpha} \tag{11a}$$

$$C_\alpha^+ = (\mathrm{i}/\sqrt{2})(X_\alpha + X_{-\alpha}) \qquad C_\alpha^- = (1/\sqrt{2})(X_\alpha - X_{-\alpha}) \qquad \tilde{H}_\alpha = -\mathrm{i}H_\alpha \tag{11b}$$

$$\delta(C_\alpha^\pm) = C_\alpha^\pm \otimes \mathrm{e}^{\tilde{H}_\alpha h_\alpha/4} + \mathrm{e}^{-\tilde{H}_\alpha h_\alpha/4} \otimes C_\alpha^\pm \qquad \alpha \in \Delta_k^+ \cap \Delta_S. \tag{11c}$$

Since $\mathcal{M}_0 = \mathcal{M}_0^s \oplus \mathcal{Z}_0^m$ is a compact reductive Lie algebra we have to choose how to do the deformation in such cases. Our choice is to preserve the reductive structure, i.e. writing in more detail $\mathcal{M}_0 = \oplus_j \mathcal{M}_0^{sj} \oplus \oplus_k \mathcal{Z}_0^{mk}$, where $\mathcal{M}_0^{sj}$ is simple and $\mathcal{Z}_0^{mk}$ is one-dimensional, then we shall have the Hopf algebra $\mathrm{U}_q(\mathcal{M}_0) = \otimes_j U_q(\mathcal{M}_0^{sj}) \otimes \otimes_k U_q(\mathcal{Z}_0^{mk})$, where we also have to specify that if $\mathcal{Z}_0^{mk}$ is spanned by $K$ then $\mathrm{U}_q(\mathcal{Z}_0^{mk})$ is spanned by $K, q^{\pm K/4}$.

*Remark 2. Formulae (8b), (11) (with $h_\alpha \in \mathbb{R}$) determine completely a Drinfeld–Jimbo q-deformation of any compact semi-simple Lie algebra* [1] (when all roots of $\Delta$ are imaginary). Here one may take $\sigma$ as an antilinear involution and coalgebra homomorphism such that $\sigma(X_\alpha^\pm) = -X_\alpha^\mp$, $\forall \alpha \in \Delta$, $\sigma(H) = -H$, $\forall H \in \mathcal{H}$. Note that in this case the $q$-deformation inherited from $\mathrm{U}_q(\mathcal{G}^\mathbb{C})$ is often used in the physics literature without the basis change (11).

Returning to the general situation, so far we have consistently chosen the generators of $\tilde{\mathcal{N}}_0^1 \oplus \mathcal{A}_0 \oplus \mathcal{M}_0 \oplus \mathcal{N}_0^1$ (cf (3)) as linear combinations of the generators of $\mathcal{H}_0 \oplus \oplus_{\alpha\in\Delta_r^0\cup\Delta_i^0}\mathcal{G}_\alpha$. Now it remains to choose consistently the generators of $\tilde{\mathcal{N}}_0^2$, $\mathcal{N}_0^2$, respectively, as linear combinations of the generators of the rest of $\mathcal{G}^\mathbb{C}$, i.e. of $\oplus_{\alpha\in\Delta_c^{0+}}\mathcal{G}_\alpha$, $\oplus_{\alpha\in\Delta_c^{0-}}\mathcal{G}_\alpha$, respectively. If $\alpha \in \Delta_c^0$, $\lambda = \alpha|_{\mathcal{A}_0}$, then $\dim_R \mathcal{G}_\lambda^0 > 1$. Let $\Delta_\lambda = \{\alpha \in \Delta | \alpha|_{\mathcal{A}_0} = \lambda\}$. If $\alpha \in \Delta_c^0$, then we have $X_\alpha = Y_\alpha + Z_\alpha$, where $Y_\alpha \in \mathcal{P}^\mathbb{C}$, $Z_\alpha \in \mathcal{K}^\mathbb{C}$. Now we can see that $\mathcal{G}_\lambda^0 = \mathrm{RLS}\{\tilde{X}_\alpha = Y_\alpha + \mathrm{i}Z_\alpha, \forall\alpha \in \Delta_\lambda\}$. The actual choice of basis in $\mathcal{G}_\lambda^0$ is a matter of convenience, (cf the examples below), and is related to the choice of $\sigma$ and $q$, and to the general property that $\mathrm{U}_q(\tilde{\mathcal{P}}_0^0)$, $\mathrm{U}_q(\mathcal{P}_0^0)$ are Hopf subalgebras of $\mathrm{U}_q(\mathcal{G})$.

### 3.2. *q-deformations with other Cartan subalgebras*

For the purposes of $q$-deformations we also need to consider Cartan subalgebras $\mathcal{H}$ which are not conjugate to $\mathcal{H}_0$. Cartan subalgebras which represent different conjugacy classes may be chosen as $\mathcal{H} = \mathcal{H}_k \oplus \mathcal{A}$, where $\mathcal{H}_k$ is compact, $\mathcal{A}$ is non-compact, $\dim \mathcal{A} < \dim \mathcal{A}_0$ if $\mathcal{H}$ is non-conjugate to $\mathcal{H}_0$. The Cartan subalgebras with maximal dimension of $\mathcal{A}$ are conjugate to $\mathcal{H}_0$; also those with a minimal dimension of $\mathcal{A}$ are conjugate to each other.

All notions introduced up until now are easily generalized for $\mathcal{H} = \mathcal{H}_k \oplus \mathcal{A}$ non-conjugate to $\mathcal{H}_0$. We note the differences and notationwise we drop all zero subscripts and superscripts. One difference is that the algebra $\mathcal{M}$ is the centralizer of $\mathcal{A}$ in $\mathcal{G}$

(mod $\mathcal{A}$) and thus is, in general, a non-compact reductive Lie algebra which has the compact $\mathcal{H}_k$ as Cartan subalgebra (besides, in general, other non-compact Cartan subalgebras); in particular, if $\mathcal{G}$ has a compact Cartan subalgebra then for the choice $\mathcal{A} = 0$ one has $\mathcal{M} = \mathcal{G}$. For the purposes of the $q$-deformation we shall use this compact Cartan subalgebra, i.e. we set $\mathcal{H}^m = \mathcal{H}_k$. Further, the classification of the roots of $\Delta$ with respect to $\mathcal{H}$ goes as before. The difference is that if $\alpha \in \Delta_i$ then $\mathcal{L}_\alpha$ may also be non-compact. Thus for $\alpha \in \Delta_i$ the root $\alpha$ is called *singular*, $\alpha \in \Delta_s$, if $\mathcal{L}_\alpha$ is non-compact, and $\alpha$ is called *compact*, $\alpha \in \Delta_k$, if $\mathcal{L}_\alpha$ is compact. Thus $\Delta_i = \Delta_s \cup \Delta_k$. Formulae (8*b*) hold for $\Delta_k$, while for $\alpha \in \Delta_s$ we have

$$\mathcal{L}_\alpha = \mathrm{RLS}\{\mathrm{i}H_\alpha, \mathrm{i}(X_\alpha - X_{-\alpha}), X_\alpha + X_{-\alpha}\} \qquad \alpha \in \Delta_s^+ \tag{12a}$$

$$[S_\alpha^+, S_\alpha^-] = \frac{\sinh(\tilde{H}_\alpha h_\alpha/2)}{\sin(h_\alpha/2)} \qquad [\tilde{H}_\alpha, S_\alpha^\pm] = \mp S_\alpha^\mp \qquad q_\alpha = q^{(\alpha,\alpha)/2} = \mathrm{e}^{-\mathrm{i}h_\alpha} \tag{12b}$$

$$S_\alpha^+ = (1/\sqrt{2})(X_\alpha + X_{-\alpha}) \qquad S_\alpha^- = (\mathrm{i}/\sqrt{2})(X_\alpha - X_{-\alpha}) \qquad \tilde{H}_\alpha = -\mathrm{i}H_\alpha \tag{12c}$$

$$\delta(S_\alpha^\pm) = S_\alpha^\pm \otimes \mathrm{e}^{\tilde{H}_\alpha h_\alpha/4} + \mathrm{e}^{-\tilde{H}_\alpha h_\alpha/4} \otimes S_\alpha^\pm \qquad \alpha \in \Delta_s^+ \cap \Delta_S. \tag{12d}$$

Furthermore as before the set of the imaginary roots in $\Delta$ may be identified with the root system of $\mathcal{M}^{s\mathbb{C}}$. Thus formulae (8*b*), (11) and (12) also give the deformation $\mathrm{U}_q(\mathcal{M}^s)$. Since the centre of $\mathcal{M}$ is compact (it is in the Cartan subalgebra $\mathcal{H}^m$ which is compact) then the deformation $\mathrm{U}_q(\mathcal{Z}^m)$ is given as after (11). Thus the Hopf algebra $\mathrm{U}_q(\mathcal{M})$ is given. Otherwise, the considerations for the factors $\mathcal{N}$, $\tilde{\mathcal{N}}$ go as for $\mathcal{N}_0$, $\tilde{\mathcal{N}}_0$.

*Thus our scheme provides a different q-deformation for each conjugacy class of Cartan subalgebras.*

### 3.3. *q-deformations for arbitrary parabolic subalgebras and reductive Lie (super)algebras*

Until now our data are the non-conjugate Cartan subalgebras $\mathcal{H} = \mathcal{H}_k \oplus \mathcal{A}$ and related with this Bruhat decompositions:

$$\mathcal{G} = \tilde{\mathcal{N}} \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N} = \tilde{\mathcal{N}}^1 \oplus \tilde{\mathcal{N}}^2 \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N}^1 \oplus \mathcal{N}^2. \tag{13}$$

In this decomposition a special role for the $q$-deformations is played by the subalgebra $\mathcal{P}_0 = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ (or equivalently by its Cartan involution conjugated $\tilde{\mathcal{P}}_0 = \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$). It is called a minimal parabolic subalgebra. A standard parabolic subalgebra is any subalgebra $\mathcal{P}'$ of $\mathcal{G}$ such that $\mathcal{P}_0 \subseteq \mathcal{P}'$. The number of standard parabolic subalgebras, including $\mathcal{P}_0$ and $\mathcal{G}$, is $2^r$, $r = \dim \mathcal{A}$. They are all of the form $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{N}' \subseteq \mathcal{N}$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}$ (mod $\mathcal{A}'$); $\mathcal{N}'$ (respectively $\tilde{\mathcal{N}}' = \theta\mathcal{N}'$) is comprised from the negative (respectively positive) root spaces of the restricted root system $\Delta'_R$ of $(\mathcal{G}, \mathcal{A}')$. One also has the analogue of (3), (13):

$$\mathcal{G} = \tilde{\mathcal{N}}' \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}'. \tag{14}$$

Note that $\mathcal{M}'$ is a non-compact reductive Lie algebra which has a non-compact Cartan subalgebra $\mathcal{H}'^m \cong \mathcal{H}_k \oplus \mathcal{H}_n$, where $\mathcal{H}_n$ is non-compact and $\mathcal{A} \cong \mathcal{H}_n \oplus \mathcal{A}'$. This Cartan subalgebra $\mathcal{H}'^m$ of $\mathcal{M}'$ will be chosen for the purposes of the $q$-deformation.

Thus we need to extend our scheme to non-compact reductive Lie algebras. Let $\hat{\mathcal{G}} = \mathcal{G} \oplus \mathcal{Z} = \hat{\mathcal{K}} \oplus \hat{\mathcal{P}}$ be a real reductive Lie algebra, where $\mathcal{G}$ is the semi-simple part of $\hat{\mathcal{G}}$, $\mathcal{Z}$ is the centre of $\hat{\mathcal{G}}$; $\hat{\mathcal{K}}, \hat{\mathcal{P}}$ are the $+1, -1$ eigenspaces of the Cartan involution $\hat{\theta}$; $\hat{\mathcal{A}}' = \mathcal{A}' \oplus \mathcal{Z}_p$, is the analogue of $\mathcal{A}'$, $\mathcal{Z}_p = \mathcal{Z} \cap \hat{\mathcal{P}}$. The root system of the pair $(\hat{\mathcal{G}}, \hat{\mathcal{A}}')$ coincides with $\Delta'_R$ and the subalgebras $\tilde{\mathcal{N}}'$ and $\mathcal{N}'$ are inherited from $\mathcal{G}$. The decomposition (3) then is

$$\hat{\mathcal{G}} = \tilde{\mathcal{N}}' \oplus \hat{\mathcal{A}}' \oplus \hat{\mathcal{M}}' \oplus \mathcal{N}' \tag{15}$$

where $\hat{\mathcal{M}}' = \mathcal{M}'^{s} \oplus \hat{\mathcal{Z}}'^{m}$, $\hat{\mathcal{Z}}'^{m} = \mathcal{Z}'^{m} \oplus \mathcal{Z} \cap \hat{\mathcal{K}}$. As in the compact reductive case we choose a deformation which preserves the splitting of $\hat{\mathcal{G}}$, i.e. $\mathrm{U}_q(\hat{\mathcal{G}}) = \mathrm{U}_q(\mathcal{G}) \otimes \mathrm{U}_q(\mathcal{Z})$, and even further into simple Lie subalgebras and one-dimensional central subalgebras.

*Remark 3.* A general property of the deformations $\mathrm{U}_q(\mathcal{G})$ obtained by the above procedure is that $\mathrm{U}_q(\mathcal{M}_0)$, $\mathrm{U}_q(\tilde{\mathcal{P}}_0)$, $\mathrm{U}_q(\mathcal{P}_0)$ are Hopf subalgebras of $\mathrm{U}_q(\mathcal{G})$.

The above scheme can be immediately applied in the case when $\mathcal{G}$ is a real form of a basic classical Lie superalgebra. This is illustrated in section 7.

## 4. Examples

### *4.1. so(p, r)*

Let $\mathcal{G} = \mathrm{so}(p, r)$, with $p \geqslant r \geqslant 2$ or $p > r \geqslant 1$ with generators: $M_{AB} = -M_{BA}$, $A, B = 1, \ldots, p+r$, $\eta_{AB} = \mathrm{diag}\ (- \cdots - + \cdots +)$, ($p$ times minus, $r$ times plus) which obey

$$[M_{AB}, M_{CD}] = \mathrm{i}(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}). \tag{16}$$

Besides the 'physical' generators $M_{AB}$ we shall also use the 'mathematical' generators $Y_{AB} = -\mathrm{i} M_{AB}$. One has $\mathcal{K} \cong \mathrm{so}(p) \oplus \mathrm{so}(r)$ if $r \geqslant 2$ and $\mathcal{K} \cong \mathrm{so}(p)$ if $r = 1$. The generators of $\mathcal{K}$ are $M_{AB}$ with $1 \leqslant A < B \leqslant p$ and $p+1 \leqslant A < B \leqslant p+r$. The split rank is equal to $r$; $\mathcal{M} \cong \mathrm{so}(p-r)$, if $p - r \geqslant 2$ and $\mathcal{M} = 0$ if $p - r = 0, 1$, dim $\tilde{\mathcal{N}} = \dim \mathcal{N} = r(p-1)$. Furthermore the dimensions of the roots in the root system $\Delta$ of $\mathrm{so}(p+r, \mathbb{C})$, and in $\Delta_R$ depending on the parity of $p + r$ are given by

| roots | $p+r$ even | $p+r$ odd | |
|---|---|---|---|
| $\lvert\Delta_r^{\pm}\rvert$ | $r(r-1)$ | $r^2$ | |
| $\lvert\Delta_i^{\pm}\rvert$ | $(p-r)(p-r-2)/4$ | $(p-r-1)^2/4$ | (17) |
| $\lvert\Delta_c^{\pm}\rvert$ | $r(p-r)$ | $r(p-r-1)$ | |
| $\lvert\Delta_R^{\pm}\rvert$ | $r^2$ | $r(r+1)$. | |

Note that the algebra $\mathrm{so}(2n+1, 1)$ has only one conjugacy class of Cartan subalgebras. Thus in these cases our $q$-deformation is unique. The algebra $\mathrm{so}(2n, 1)$ has two conjugacy classes of Cartan subalgebras and in these cases there are two $q$-deformations which we illustrate below for $n = 1$.

### 4.2. q-deformed so(2,1)

Using notation from above $A, B = 1, 2, 0$, $(- - +)$; $Y_{12}$ is the generator of $\mathcal{K}$ and we may choose $Y_{20}$ for the generator of $\mathcal{A}$; $\mathcal{M} = 0$. Thus we can choose either $Y_{20}$ or $Y_{12}$ as a generator of $\mathcal{H}$ and of $\mathcal{H}^{\mathbb{C}}$. Let $\Delta^{\pm} = \{\pm\alpha\}$ be the root system of $\mathcal{G}^{\mathbb{C}} = \mathrm{sl}(2, \mathbb{C})$. If $\mathcal{H}^{\mathbb{C}}$ is generated by $Y_{20}$ (and $\mathcal{H} = \mathcal{H}_0 = \mathcal{A}$) then $\alpha$ is a real root and this deformation, denoted $\mathrm{U}_q^0(\mathrm{so}(2,1))$, is given by formulae (9) and (7) over $\mathbb{R}$. If $\mathcal{H}^{\mathbb{C}}$ is generated by $Y_{12}$ then $\alpha$ is a singular imaginary root and the deformation, denoted $\mathrm{U}_q^1(\mathrm{so}(2,1))$, is given by formulae (12) with $h_\alpha \in \mathbb{R}$.

### 4.3. q-deformed Lorentz algebra $U_q(so(3,1))$

With $A, B = 1, 2, 3, 0$, $(- - - +)$, choose $\tilde{D} = M_{30}$ for the generator of $\mathcal{A}$ and $H = M_{12}$ for the generator of $\mathcal{M}$. From the above table we see that all roots are complex (as is also verified by a simple calculation). It is convenient to use the generators $M^{\pm} = -M_{23} \pm \mathrm{i}M_{13} \in \mathcal{K}^{\mathbb{C}}$, $N^{\pm} = -M_{10} \mp \mathrm{i}M_{20} \in \mathcal{P}^{\mathbb{C}}$. We recall that $\mathcal{G}^{\mathbb{C}} = \mathrm{so}(4, \mathbb{C}) \cong \mathrm{so}(3, \mathbb{C}) \oplus \mathrm{so}(3, \mathbb{C})$. The generators of the two commuting $\mathrm{so}(3, \mathbb{C})$ algebras are $X_1^{\pm}, H_1$ and $X_2^{\pm}, H_2$, where

$$X_1^{\pm} = (1/2)(M^{\pm} - \mathrm{i}N^{\pm}) \qquad H_1 = H - \mathrm{i}\tilde{D}$$
$$X_2^{\pm} = (1/2)(M^{\pm} + \mathrm{i}N^{\pm}) \qquad H_2 = H + \mathrm{i}\tilde{D}. \tag{18}$$

We use $\mathrm{U}_q(\mathrm{so}(4, \mathbb{C})) = U_q(\mathrm{so}(3, \mathbb{C})) \otimes U_q(\mathrm{so}(3, \mathbb{C}))$ given by

$$[X_a^+, X_a^-] = [H_a] \qquad [H_a, X_a^{\pm}] = \pm 2X_a^{\pm} \qquad a = 1, 2 \tag{19}$$

and the Hopf algebra structure is given just by (7) replacing $X_i$ with $X_a$. Using this we obtain the following $\mathrm{U}_q(\mathrm{so}(3,1))$ relations with $q = \mathrm{e}^h \in \mathbb{R}$:

$$[H, M^{\pm}] = \pm M^{\pm} \qquad [H, N^{\pm}] = \pm N^{\pm}$$
$$[\tilde{D}, M^{\pm}] = \pm N^{\pm} \qquad [\tilde{D}, N^{\pm}] = \mp M^{\pm} \tag{20a}$$

$$[M^+, M^-] = [N^-, N^+] = 2[H]\cos(\tilde{D}h/2)$$
$$[M^{\pm}, N^{\mp}] = \pm 2\frac{\cosh(Hh/2)\sin(\tilde{D}h/2)}{\sinh(h/2)} \tag{20b}$$

$$\delta(M^{\pm}) = M^{\pm} \otimes \mathrm{e}^{Hh/4}\cos(\tilde{D}h/4) - N^{\pm} \otimes \mathrm{e}^{Hh/4}\sin(\tilde{D}h/4)$$
$$+ \mathrm{e}^{-Hh/4}\cos(\tilde{D}h/4) \otimes M^{\pm} + \mathrm{e}^{-Hh/4}\sin(\tilde{D}h/4) \otimes N^{\pm} \tag{21a}$$

$$\delta(N^{\pm}) = N^{\pm} \otimes \mathrm{e}^{Hh/4}\cos(\tilde{D}h/4) + M^{\pm} \otimes \mathrm{e}^{Hh/4}\sin(\tilde{D}h/4)$$
$$+ \mathrm{e}^{-Hh/4} cos(\tilde{D}h/4) \otimes N^{\pm} - \mathrm{e}^{-Hh/4}\sin(\tilde{D}h/4) \otimes M^{\pm} \tag{21b}$$

*4.4. q-deformed so(4,1) and so(3,2)*

*4.4.1.* The algebras so(4, 1) and so(3, 2) have the same complexification $\mathcal{G}^{\mathbb{C}} =$ so$(5, \mathbb{C})$. The root system of so$(5, \mathbb{C})$ is given by $\Delta^{\pm} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\}$; the simple roots are $\alpha_1, \alpha_2$, while $\alpha_3 = \alpha_1 + \alpha_2,\ \alpha_4 = 2\alpha_1 + \alpha_2$; the products between the simple roots are: $(\alpha_1, \alpha_1) = 2 = -(\alpha_1, \alpha_2)$, $(\alpha_2, \alpha_2) = 4$. The Cartan–Weyl basis for the non-simple roots is given by (cf [12])

$$X_3^+ = X_1^+ X_2^+ - q^{(\alpha_1,\alpha_2)/2} X_2^+ X_1^+ = X_1^+ X_2^+ - q^{-1} X_2^+ X_1^+ \equiv [X_1^+, X_2^+]_{q^{-1}} \qquad (22a)$$

$$X_3^- = X_2^- X_1^- - q^{-(\alpha_1,\alpha_2)/2} X_1^- X_2^- = X_2^- X_1^- - q X_1^- X_2^- = [X_2^-, X_1^-]_q \qquad (22b)$$

$$X_4^+ = X_1^+ X_3^+ - q^{(\alpha_1,\alpha_3)/2} X_3^+ X_1^+ = X_1^+ X_3^+ - X_3^+ X_1^+ , X_4^- = X_3^- X_1^- - X_1^- X_3^- . \qquad (22c)$$

All other commutation relations follow from these definitions. We shall mention only

$$[X_4^{\pm}, X_2^{\pm}] = \pm(q^{\pm 1} - 1)(X_3^{\pm})^2 \qquad [X_4^{\pm}, X_2^{\mp}] = \pm(1 - q^{-2})(1 - q^{\pm 1})(X_1^{\pm})^2 q^{\pm H_2} . \qquad (23)$$

*4.4.2.* Let $\mathcal{G} =$ so(4, 1). With $A, B = 1, 2, 3, 4, 0$ $(- - - - +)$, choose $Y_{30}$ for the generator of $\mathcal{A}$; $\mathcal{M} \cong$ so(3) with generators $Y_{ab}$, $a, b = 1, 2, 4$ and we choose $Y_{12}$ for the generator of its Cartan subalgebra. The algebra $\mathcal{G} =$ so(4, 1) has two non-conjugate Cartan subalgebras; besides $\mathcal{H}_0$ generated by $Y_{30}, Y_{12}$ we have a compact Cartan subalgebra $\mathcal{H}_1$ generated, say, by $Y_{12}, Y_{34}$.

In the case of $\mathcal{H} = \mathcal{H}_0$ the generators of $\mathcal{G}$ are expressed in terms of those of so$(5, \mathbb{C})$ by

$$Y_{30} = -H_1, Y_{12} = \mathrm{i}(H_1 + H_2) \qquad Y_{14} = (1/\sqrt{2})(X_3^+ + X_3^-) \qquad (24a)$$

$$Y_{24} = (\mathrm{i}/\sqrt{2})(X_3^- - X_3^+) \qquad Y_{34} = (1/\sqrt{2})(X_1^+ + X_1^-)$$

$$Y_{40} = (1/\sqrt{2})(X_1^+ - X_1^-) \qquad (24b)$$

$$Y_{13} = (1/2)(X_2^- + X_2^+ + X_4^+ + X_4^-) \qquad Y_{23} = (\mathrm{i}/2)(X_2^- - X_2^+ - X_4^+ + X_4^-) \qquad (24c)$$

$$Y_{10} = (1/2)(X_2^- - X_2^+ + X_4^+ - X_4^-) \qquad Y_{20} = (\mathrm{i}/2)(X_2^- + X_2^+ - X_4^+ + X_4^-). \qquad (24d)$$

Now we can give all commutaion relations and Hopf algebra operations for $Y_{AB}$ as generators of $q$-deformed so(4, 1) as inherited from $\mathrm{U}_q(\mathrm{so}(5, \mathbb{C}))$. The deformation obtained in this way is denoted by $\mathrm{U}^0_{41}$.

In the case of the Cartan subalgebra $\mathcal{H}_1$ we have: $Y_{34} = \mathrm{i}H_1$, $Y_{12} = -\mathrm{i}(H_1 + H_2)$. For the lack of space we omit the other generators. The deformation obtained in this way we denote by $\mathrm{U}^1_{41}$.

*4.4.3.* Let $\mathcal{G} = \mathrm{so}(3,2)$. With $A, B = 1,2,3,4,0$ $(---++)$, choose $Y_{20}$ and $Y_{34}$ as generators of $\mathcal{H}_0 = \mathcal{A}$. The algebra $\mathcal{G} = \mathrm{so}(3,2)$ has three non-conjugate Cartan subalgebras; besides $\mathcal{H}_0$ we have $\mathcal{H}_1$ generated, say, by $Y_{12}, Y_{30}$ and $\mathcal{H}_2$ generated, say, by $Y_{12}, Y_{40}$. Thus $\mathcal{H}_a, a = 0,1,2$, is a Cartan subalgebra with $a$ compact generators.

For the Cartan subalgebra $\mathcal{H}_0$ we identify $Y_{34} = H_1$, $Y_{20} = H_1 + H_2$; for $\mathcal{H}_1$ we have $Y_{12} = -\mathrm{i}H_1$, $Y_{34} = H_1 + H_2$; for $\mathcal{H}_2$ one uses $M_{12} = H_1$, $M_{40} = H_1 + H_2$. (The last deformation was used in [14].) We shall denote the deformation using the Cartan subalgebra $\mathcal{H}_a$ by $\mathrm{U}^a_{32}$.

## 5. $q$-deformed conformal algebra $\mathrm{U}_q(\mathrm{su(2,2)})$

The root system of the complexification $\mathrm{sl}(4,\mathbb{C})$ of $\mathrm{su}(2,2)$ is given by $\Delta^\pm = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_{12}, \pm\alpha_{23}, \pm\alpha_{13}\}$; the simple roots are $\alpha_1, \alpha_2, \alpha_3$, while $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$, $\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3$; all roots are of length 2 and the non-zero products between the simple roots are: $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1$. The Cartan–Weyl basis for the non-simple roots is given by (cf [2, 15])

$$X^\pm_{jk} = \pm q^{\mp 1/4}(q^{1/4}X^\pm_j X^\pm_k - q^{-1/4}X^\pm_k X^\pm_j) \qquad (jk) = (12), (23) \tag{25a}$$

$$X^\pm_{13} = \pm q^{\mp 1/4}(q^{1/4}X^\pm_1 X^\pm_{23} - q^{-1/4}X^\pm_{23}X^\pm_1) = \pm q^{\mp 1/4}(q^{1/4}X^\pm_{12}X^\pm_3 - q^{-1/4}X^\pm_3 X^\pm_{12}). \tag{25b}$$

All other commutation relations follow from these definitions. Besides those in (6) we have $(X^\pm_{aa} \equiv X^\pm_a)$

$$[X^+_a, X^-_{ab}] = -q^{H_a/2}X^-_{a+1b} \qquad [X^+_b, X^-_{ab}] = X^-_{ab-1}q^{-H_b/2} \qquad 1 \leqslant a < b \leqslant 3 \tag{26a}$$

$$[X^-_a, X^+_{ab}] = X^+_{a+1b}q^{-H_a/2} \qquad [X^-_b, X^+_{ab}] = -q^{H_b/2}X^+_{ab-1} \qquad 1 \leqslant a < b \leqslant 3 \tag{26b}$$

$$X^\pm_a X^\pm_{ab} = q^{1/2}X^\pm_{ab}X^\pm_a \qquad 1 \leqslant a < b \leqslant 3 \tag{26c}$$

$$[X^\pm_2, X^\pm_{13}] = 0 \qquad [X^\pm_2, X^\mp_{13}] = 0 \qquad [X^+_{12}, X^-_{13}] = -q^{H_1+H_2}X^-_3 \tag{26d}$$

$$\begin{gathered}[X^-_{12}, X^+_{13}] = X^+_3 q^{-H_1-H_2} \qquad [X^+_{23}, X^-_{13}] = X^-_1 q^{-H_2-H_3} \\ [X^-_{23}, X^+_{13}] = -q^{H_2+H_3}X^+_1\end{gathered} \tag{26e}$$

$$[X^\pm_{12}, X^\pm_{23}] = \tilde{\lambda}X^\pm_2 X^\pm_{13} \qquad [X^\pm_{12}, X^\mp_{23}] = -\tilde{\lambda}q^{\pm H_2/2}X^\pm_1 X^\mp_3 \qquad \tilde{\lambda} \equiv q^{1/2} - q^{-1/2}. \tag{26f}$$

Let $\mathcal{G} = \mathrm{su}(2,2) \cong \mathrm{so}(4,2)$. It has three non-conjugate classes of Cartan subalgebras represented, say, by $\mathcal{H}^a$, $a = 0,1,2$ with $a$ non-compact generators. Thus, according to our procedure, it has five different deformations—three in the case of $\mathcal{H}^2$ (since there are three non-trivial parabolic subalgebras) and one each for the other two choices of Cartan subalgebras. We shall work with the most non-compact Cartan subalgebra $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}^2$ and with the maximal parabolic subalgebra.

Using the notation from section 4.1. with $A, B = 1,2,3,5,6,0$ $(----++)$, choose $Y_{30}$ and $Y_{56}$ as generators of $\mathcal{A}$ and $Y_{12}$ for the generator of $\mathcal{M}$. Since su(2,2) is the conformal algebra of four-dimensional Minkowski spacetime we would like to deform it consistently with the subalgebra structure relevant for the physical applications. These subalgebras are the *Lorentz* subalgebra $\mathcal{M}' \cong \mathrm{so}(3,1)$ generated by $Y_{\mu\nu}$, $\mu,\nu = 1,2,3,0$, the subalgebra $\tilde{\mathcal{N}}'$ of *translations* generated by $P_\mu = Y_{\mu 5} + Y_{\mu 6}$; the subalgebra $\mathcal{N}'$ of *special conformal transformations* generated by $K_\mu = Y_{\mu 5} - Y_{\mu 6}$, the *dilatations* subalgebra $\mathcal{A}'$ generated by $D = Y_{56}$. The commutation relations besides those for the Lorentz subalgebra are

$$[D, Y_{\mu\nu}] = 0 \qquad [D, P_\mu] = P_\mu \qquad [D, K_\mu] = -K_\mu \tag{27a}$$

$$[Y_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu \qquad [Y_{\mu\nu}, K_\lambda] = \eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu$$
$$[P_\mu, K_\nu] = 2Y_{\mu\nu} + 2\eta_{\mu\nu} D. \tag{27b}$$

The algebra $\mathcal{P}_{\max} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$ (or equivalently $\tilde{\mathcal{P}}_{\max} = \mathcal{M}' \oplus \mathcal{A}' \oplus \tilde{\mathcal{N}}'$) is the so called *maximal parabolic subalgebra* of $\mathcal{G}$, where $\tilde{\mathcal{N}}'$, respectively, $\mathcal{N}'$, is the root vector space of the restricted root system $\Delta'_R = \{\pm\lambda; \lambda(D) = 1\}$ of $(\mathcal{G}, \mathcal{A}')$, corresponding to $\lambda$, respectively, $-\lambda$, (cf subsection 3.3).

For the Lorentz algebra generators we have the following expressions (which are inverse to (18)):

$$\begin{aligned} H &= -Y_{30} = (1/2)(H_1 + H_3) \qquad & M^\pm &= -\mathrm{i}Y_{13} \pm \mathrm{i}Y_{10} = X_1^\pm + X_3^\pm \\ \tilde{D} &= -Y_{12} = (\mathrm{i}/2)(H_1 - H_3) \qquad & N^\pm &= -\mathrm{i}Y_{20} \pm \mathrm{i}Y_{23} = \mathrm{i}(X_1^\pm - X_3^\pm). \end{aligned} \tag{28}$$

For the dilatations, translations and special conformal transformations we have

$$D = (1/2)(H_1 + H_3) + H_2 \tag{29}$$

$$\begin{aligned} P_0 &= \mathrm{i}(X_{13}^+ + X_2^+) \qquad & P_1 &= \mathrm{i}(X_{12}^+ + X_{23}^+) \\ P_2 &= X_{12}^+ - X_{23}^+ \qquad & P_3 &= \mathrm{i}(X_2^+ - X_{13}^+) \end{aligned} \tag{30}$$

$$\begin{aligned} K_0 &= -\mathrm{i}(X_{13}^- + X_2^-) \qquad & K_1 &= \mathrm{i}(X_{12}^- + X_{23}^-) \\ K_2 &= X_{23}^- - X_{12}^- \qquad & K_3 &= \mathrm{i}(X_2^- - X_{13}^-). \end{aligned} \tag{31}$$

Now we can derive the relations in $U_q(\mathrm{su}(2,2))$:

(i) According to our general scheme the deformed Lorentz subalgebra is a Hopf subalgebra; its deformation is described by formulae (20) and (21).

(ii) The commutation relations of the generators $H, \tilde{D}, D$ of the Cartan subalgebra $\mathcal{H} = \mathcal{H}_0$ are not deformed.

(iii) The deformation of the translations and special conformal transformations subalgebras is given by

$$\begin{aligned} &P_a(P_1 \pm \mathrm{i}P_2) = q^{\mp 1/2}(P_1 \pm \mathrm{i}P_2)P_a \qquad a = 0,3 \\ &[P_0, P_3] = 0 \qquad [P_1 + \mathrm{i}P_2, P_1 - \mathrm{i}P_2] = \tilde{\lambda}(P_0^2 - P_3^2) \end{aligned} \tag{32}$$

$$\begin{aligned} &K_a(K_1 \pm \mathrm{i}K_2) = q^{\pm 1/2}(K_1 \pm \mathrm{i}K_2)K_a \qquad a = 0,3 \\ &[K_0, K_3] = 0 \qquad [K_1 + \mathrm{i}K_2, K_1 - \mathrm{i}K_2] = \tilde{\lambda}(K_0^2 - K_3^2) \end{aligned} \tag{33}$$

(iv) The commutation relations of $M^\pm$ with $P_\mu$ are given by

$$M^+(P_1 - \mathrm{i}P_2) - q^{-1/2}(P_1 - \mathrm{i}P_2)M^+ = P_0 - P_3$$

$$M^+(P_1 + \mathrm{i}P_2) - q^{1/2}(P_1 + \mathrm{i}P_2)M^+ = q^{1/2}(P_0 - P_3)$$

$$M^+(P_0 - P_3) - \frac{[2]}{2}(P_0 - P_3)M^+ = \frac{\mathrm{i}\tilde{\lambda}}{2}(P_0 - P_3)N^+ \tag{34}$$

$$M^+(P_0 + P_3) - \frac{[2]}{2}(P_0 + P_3)M^+ = \frac{\mathrm{i}\tilde{\lambda}}{2}(P_0 + P_3)N^+ + (P_1 + \mathrm{i}P_2) - q^{1/2}(P_1 - \mathrm{i}P_2)$$

$$[M^-, P_1 - \mathrm{i}P_2] = -q^{(\mathrm{i}\tilde{D}+H)/2}(P_0 + P_3)$$

$$[M^-, P_1 + \mathrm{i}P_2] = (P_0 + P_3)q^{(\mathrm{i}\tilde{D}-H)/2} \tag{35a}$$

$$[M^-, P_0 - P_3] = (P_1 - \mathrm{i}P_2)q^{(\mathrm{i}\tilde{D}-H)/2} - q^{(\mathrm{i}\tilde{D}+H)/2}(P_1 + \mathrm{i}P_2)$$

$$[M^-, P_0 + P_3] = 0. \tag{35b}$$

The commutation relations between $M^\pm$ and $K_\mu$ are obtained from the above by the following changes: $M^\pm \mapsto M^\mp$, $N^+ \mapsto -N^-$, $H \mapsto -H$, $\tilde{D} \mapsto \tilde{D}$, $P_\mu \mapsto \eta_{\mu\mu} K_\mu$, $q^{1/2} \mapsto q^{-1/2}$. These follow from the automorphism of $\mathrm{U}_q(\mathcal{G}^{\mathbb{C}})$: $X_1^\pm \longleftrightarrow X_3^\mp$, $H_1 \longleftrightarrow -H_3$, $X_2^\pm \mapsto -X_2^\mp$, $H_2 \mapsto -H_2$, $q^{1/2} \mapsto q^{-1/2}$ (then $X_{12}^\pm \longleftrightarrow -X_{23}^\mp$, $X_{13}^\pm \mapsto -X_{13}^\mp$). The commutation relations between $N^\pm$ and $P_\mu, K_\mu$ are obtained from those between $M^\pm$ and $P_\mu$ by the changes $M^\pm \longleftrightarrow \mathrm{i}N^\pm$, $P_0 \longleftrightarrow -P_3$, $P_1 \longleftrightarrow -\mathrm{i}P_2$ and from those between $M^\pm$ and $K_\mu$ by the changes $M^\pm \longleftrightarrow -\mathrm{i}N^\pm$, $K_0 \longleftrightarrow K_3$, $K_1 \longleftrightarrow -\mathrm{i}K_2$.

(v) For $[P_\mu, K_\nu]$ we have

$$[P_1 \pm \mathrm{i}P_2, K_1 \pm \mathrm{i}K_2] = \pm\tilde{\lambda} q^{\mp(H-D)/2}(M^+ \pm \mathrm{i}N^+)(M^- \mp \mathrm{i}N^-)$$

$$[P_1 \pm \mathrm{i}P_2, K_1 \mp \mathrm{i}K_2] = 4[\pm\mathrm{i}\tilde{D} - D] \tag{36a}$$

$$[P_0 \pm P_3, K_3 \mp K_0] = \pm 4[\pm H - D]$$

$$[P_0 \pm P_3, K_3 \pm K_0] = 0 \tag{36b}$$

$$[P_1 - \mathrm{i}P_2, K_3 - K_0] = 2(M^+ - \mathrm{i}N^+)q^{(H-D)/2}$$

$$[P_1 - \mathrm{i}P_2, K_0 + K_3] = 2(M^- + \mathrm{i}N^-)q^{-(D+\mathrm{i}\tilde{D})/2} \tag{36c}$$

$$[P_1 + \mathrm{i}P_2, K_3 - K_0] = -2q^{(D-H)/2}(M^+ + \mathrm{i}N^+)$$

$$[P_1 + \mathrm{i}P_2, K_0 + K_3] = -2q^{(D-\mathrm{i}\tilde{D})/2}(M^- - \mathrm{i}N^-) \tag{36d}$$

and four more relations which are obtained from ($36c,d$) by the first set of changes described after formulae (35) and by $D \mapsto -D$.

The comultiplication for the Lorentz subalgebra is given by (21), for the dilatation generator $D \in \mathcal{H} \subset \mathcal{H}^{\mathbb{C}}$ it is trivial and for the translations and special conformal transformations we have

$$\delta(T^\pm) = \begin{cases} T^\pm \otimes q^{(D\pm i\tilde{D})/4} + q^{-(D\pm i\tilde{D})/4} \otimes T^\pm + \delta_1(T^\pm) \\ \quad T^\pm = P_1 \mp iP_2 \qquad K_1 \pm iK_2 \\ T^\pm \otimes q^{(D\pm H)/4} + q^{-(D\pm H)/4} \otimes T^\pm + \delta_1(T^\pm) \\ \quad T^\pm = P_0 \mp P_3 \qquad K_3 \pm K_0 \end{cases} \tag{37a}$$

$$\delta_1(T^\pm) = \begin{cases} \pm(\tilde{\lambda}/2)(M^\pm \mp iN^\pm) q^{(H-D)/4} \otimes q^{(H\pm i\tilde{D})/4} \tilde{T}^\pm \\ \quad T^+ = P_1 - iP_2 \qquad T^- = K_1 - iK_2 \\ \pm(\tilde{\lambda}/2)\tilde{T}^\pm q^{(-H\pm i\tilde{D})/4} \otimes q^{(D-H)/4}(M^\pm \mp iN^\pm) \\ \quad T^+ = P_1 + iP_2 \qquad T^- = K_1 + iK_2 \end{cases} \tag{37b}$$

$$\begin{aligned} \delta_1(T^\pm) = (\tilde{\lambda}/2)(&\tilde{T}'^\pm q^{(-H\pm i\tilde{D})/4} \otimes q^{(D\pm i\tilde{D})/4}(M^\pm \pm iN^\pm) \\ &+ (M^\pm \mp iN^\pm) q^{(-D\pm i\tilde{D})/4} \otimes q^{(H\pm i\tilde{D})/4} \tilde{T}''^\pm) \\ &T^+ = P_0 - P_3 \qquad T^- = K_0 + K_3 \\ &\tilde{T}'^+ = P_1 - iP_2 \qquad \tilde{T}'^- = K_1 - iK_2 \\ &\tilde{T}''^+ = -(P_1 + iP_2) \qquad \tilde{T}''^- = K_1 + iK_2 \end{aligned} \tag{37c}$$

$\delta_1(\tilde{T}^\pm) = 0$, $\tilde{T}^+ = P_0 + P_3$, $\tilde{T}^- = K_3 - K_0$. Consistently with the general scheme (cf remark 3) formulae (37) tell us that the deformed subalgebras of translations and special conformal transformations are not Hopf subalgebras of $\mathcal{G}$.

## 6. $q$-deformed Poincaré and Weyl algebras

The Poincaré algebra is not a semi-simple (or reductive) Lie algebra and our procedure is not directly applicable. One may try to use the fact that it is a subalgebra of the conformal algebra. Indeed, there is a $q$-deformed Poincaré algebra with generators $M^\pm$, $N^\pm$, $H$, $\hat{D} = i\tilde{D}$, $P_\mu$, and with commutations relations given by (20), (32), (34), (35) and those obtained from the latter two by the changes $M^\pm \longleftrightarrow iN^\pm$, $P_0 \longleftrightarrow -P_3$, $P_1 \longleftrightarrow -iP_2$. However, from formulae (37) it follows that the deformation of the Poincaré subalgebra of su(2,2) is not a Hopf subalgebra, but the deformation $U_q(\tilde{\mathcal{P}}_{\max})$ of the 11-generator Weyl subalgebra = Poincaré and dilatations = $\tilde{\mathcal{P}}_{\max}$—is a Hopf subalgebra of $U_q(\mathcal{G})$. Another Weyl algebra conjugate to this is $U_q(\mathcal{P}_{\max})$ with generators $M^\pm$, $N^\pm$, $H$, $\hat{D} = i\tilde{D}$, $K_\mu$, $D$ and with commutations relations given by (20), (33), and those obtained from (34), (35) as explained in the text thereafter.

Other deformed Poincaré algebras may be obtained from the contraction of $U^a_{41}$ and $U^a_{32}$ discussed in section 4.4. Only for $U^0_{41}$ and $U^1_{32}$ one may expect to obtain a deformed Lorentz subalgebra as a Hopf subalgebra after contracting $Y_{4\mu} \longrightarrow RP_\mu$, $R \longrightarrow \infty$, since $Y_{4\mu}$ are not Cartan generators. However, if $q \neq 1$, this limit is not consistent with the commutation relations which are inherited from relations (23). The other possibilty is to make contractions which involve Cartan generators. This may be a non-compact generator which is possible for $U^0_{41}$ and $U^a_{32}$, $a = 0, 1$, or a compact generator which is possible for $U^a_{41}$, $a = 0, 1$, and $U^2_{32}$. (The last case was studied in [14].) The resulting deformed Poincaré algebras will have a non-compact Hopf subalgebra in the case $U^0_{32}$ and in one of the $U^0_{41}$ cases and a compact Hopf subalgebra in the other four cases.

## 7. $q$-deformed conformal superalgebras $U_q(su(2,2/N))$

### 7.1. $q$-deformed complex superalgebras

Let $\mathcal{G}_c$ be a complex Lie superalgebra with a symmetrizable Cartan matrix $A = (a_{jk}) = A^d A^s$, where $A^s = (a^s_{jk})$ is a symmetric matrix, and $A^d = \mathrm{diag}\,(d_1, \ldots, d_n)$, $d_j > 0$. Then the $q$-deformation $U_q(\mathcal{G}_c)$ of the universal enveloping algebras $U(\mathcal{G}_c)$ is defined [12, 16] as the associative algebra over $\mathbb{C}$ with generators $X^\pm_j$, $H_j$, $j \in J = \{1, \ldots, \ell\}$ and with relations:

(i) (4*a*) with $a_{jk}$ replaced by $a^s_{jk}$ and $[\,,]$ being the supercommutator: $[Y, Z] \equiv YZ - (-1)^{\deg Y \deg Z} ZY$, $\deg H_j = \bar{0}$, $j \in J$, $\deg X^\pm_j = \bar{0}$, $j \notin \mathcal{T}$, $\deg X^\pm_j = \bar{1}$, $j \in \mathcal{T}$, $\mathcal{T} \subset J$ enumerates the set of *odd* simple roots, $J \backslash \mathcal{T}$—the set of *even* simple roots;

(ii)

$$(\mathrm{ad}_{q^\kappa} X^\pm_j)^{n_{jk}}(X^\pm_k) = 0 \qquad \text{for } j \neq k, \kappa = \pm \tag{38a}$$

(iii) [17–21], for every three simple roots, say, $\alpha_j, \alpha_{j\pm1}$, such that $(\alpha_j, \alpha_j) = 0$, $(\alpha_{j\pm1}, \alpha_{j\pm1}) \neq 0$, $(\alpha_{j+1}, \alpha_{j-1}) = 0$, $(\alpha_j, \alpha_{j+1} + \alpha_{j-1}) = 0$, the following also holds

$$[[X^\pm_j, X^\pm_{j-1}]_{q^\kappa}, [X^\pm_j, X^\pm_j]_{q^\kappa}]_{q^\kappa} = 0 \tag{38b}$$

where

$$n_{jk} = \begin{cases} 1 & \text{if } a^s_{jj} = a^s_{jk} = 0 \\ 2 & \text{if } a^s_{jj} = 0, a^s_{jk} \neq 0 \\ 1 - 2a^s_{jk}/a^s_{jj} & \text{if } a^s_{jj} \neq 0 \end{cases} \tag{39}$$

in (38*a*, *b*) one uses the deformed supercommutator:

$$(\mathrm{ad}_{q^\kappa} X^\pm_j)(X^\pm_k) = [X^\pm_j, X^\pm_k]_{q^\kappa} \equiv X^\pm_j X^\pm_k - (-1)^{\deg X^\pm_j \deg X^\pm_k} q^{\kappa(\alpha_j, \alpha_k)/2} X^\pm_k X^\pm_j. \tag{40}$$

When $\mathcal{T} = \emptyset$ relations (38*a*) for $\kappa = 1$ are the same as for $\kappa = -1$ and coincide with (4*b*). The necessity of the extra relations (iii) was communicated to the author in May 1991 independently by M Scheunert [17], V G Kac [18] and D A Leites [19]. These relations were written first for $U_q(sl(M/N; \mathbb{C}))$ in [17] (cf also [20]); then for $U_q(osp(M/2N))$ in [20]; here they are given as in [21].

The Hopf algebra structure is given by formulae (7), however, with $\rho = \rho_0 - \rho_1$, $\rho_s = \frac{1}{2}\sum_{\alpha \in \Delta^+_{(s)}} \alpha$, $\Delta^+_{(0)}$ $(\Delta^-_{(1)})$ is the set of even (odd) positive roots.

Let $\mathcal{G}_c = \mathcal{G}^{\mathbb{C}} = sl(M/N; \mathbb{C})$, $\ell = M + N$. We choose a Cartan matrix with elements: $a_{jj} = a^s_{jj} = 2(1 - \delta_{jM})$, $a_{jj\pm1} = a^s_{jj\pm1} = -1$ except for $a_{MM+1} = 1$, all other elements are zero; $d_j = 1, j \leqslant M$, $d_j = -1, j > M$. Consistently the products between the simple roots are: $(\alpha_j, \alpha_j) = 2, 0, -2$ for $j < M, j = 0, (j > M)$ $(\alpha_j, \alpha_{j+1}) = -1, 1$ for $j \leqslant M (j > M)$ respectively, all other products are zero. The root system is given by

$$\Delta^{\pm} = \{\pm\alpha_{jk} = \pm(\alpha_j + \alpha_{j+1} + \cdots + \alpha_k) \,|1 \leqslant j < k \leqslant \ell, \alpha_{jj+1} = \alpha_j\}.$$

The roots $\pm\alpha_{jk}$ with $1 \leqslant j \leqslant M, M < k \leqslant \ell$ are odd, the rest are even. The Cartan–Weyl generators corresponding to non-simple roots are defined inductively in analogy to (25) (see also [12]):

$$X^+_{jk} \equiv X^+_j X^+_{j+1k} - (-1)^{\deg X^+_j \deg X^+_{j+1k}} q^{(\alpha_j, \alpha_{j+1k})/2} X^+_{j+1k} X^+_j \qquad j < k-1 \quad (41a)$$

$$X^-_{jk} \equiv X^-_{j+1k} X^-_j - (-1)^{\deg X^-_j \deg X^-_{j+1k}} q^{-(\alpha_j, \alpha_{j+1k})/2} X^-_j X^-_{j+1k} \qquad j < k-1 \quad (41b)$$

$X^{\pm}_{jj+1} \equiv X^{\pm}_j$.

Note that $\mathrm{sl}(M/M;\mathbb{C})$ is a reductive Lie superalgebra with its centre generated by $Z_M \equiv H_1 - H_{2M-1} + 2(H_2 - H_{2M-2}) + \cdots + (M-1)(H_{M-1} - H_{M+1}) + MH_M$.

### 7.2. $U_q(su(2,2/N))$

The Lie superalgebra $\mathcal{G}^S \equiv \mathrm{su}(2,2/N)$ [11] is a real non-compact form of $\mathcal{G}^{\mathbb{C}} = \mathrm{sl}(4/N;\mathbb{C})$ with Cartan decomposition and splitting into even and odd parts: $\mathcal{G}^S = \mathcal{K}^S + \mathcal{P}^S = \mathcal{G}^S_{(0)} + \mathcal{G}^S_{(1)}$ such that $\mathcal{G}^S_{(0)} \cong \mathrm{su}(2,2) \oplus u(1) \oplus \mathrm{su}(N)$, $\mathcal{K}^S_{(0)} \cong u(2) \oplus u(2) \oplus \mathrm{su}(N)$, $\dim_R \mathcal{P}^S_{(0)} = 8$, $\dim_R \mathcal{K}^S_{(1)} = \dim_R \mathcal{P}^S_{(1)} = 4N$.

The parabolic subalgebras of $\mathcal{G}^S$ are determined by the parabolic subalgebras of the non-compact subalgebra su(2,2) of the even part $\mathcal{G}^S_{(0)}$. As for su(2,2) in the present paper we consider only $q$-deformations of $\mathcal{G}$ which are consistent with the maximal parabolic subalgebra $\mathcal{P}^S_{\max} = \mathcal{M}^S \oplus \mathcal{A}^S \oplus \mathcal{N}^S$, where

$$\mathcal{A}^S = \mathcal{A}^S_{(0)} = \mathrm{LS}\{D\} \cong \mathcal{A}' \qquad \mathcal{M}^S = \mathcal{M}^S_{(0)} \cong \mathcal{M}' \oplus u(1) \oplus \mathrm{su}(N)$$
$$\mathcal{M}' \cong \mathrm{so}(3,1) \quad (42a)$$

$$\mathcal{N}^S = \mathcal{G}^-_1 \oplus \mathcal{G}^-_2 \qquad \mathcal{G}^-_k \equiv \mathcal{G}_{-\lambda_k} \qquad \lambda_1(D) = 1/2 \qquad \lambda_2 = 2\lambda_1$$
$$\dim \mathcal{G}^-_1 = 4N \qquad \mathcal{N}^S_{(0)} = \mathcal{G}^-_2 \cong \mathcal{N}' \quad (42b)$$

$$\tilde{\mathcal{N}}^S = \mathcal{G}^+_1 \oplus \mathcal{G}^+_2, \mathcal{G}^+_k \equiv \mathcal{G}_{\lambda_k} = \theta\mathcal{G}^-_k \qquad \tilde{\mathcal{N}}^S_{(0)} = \mathcal{G}^+_2 \cong \tilde{\mathcal{N}}' \quad (42c)$$

where the primed objects are su(2,2) subalgebras. The Cartan subalgebra $\mathcal{H}^S \subset \mathcal{G}^S_{(0)}$ is chosen as follows

$$\mathcal{H}^S = \mathcal{H} \oplus u(1) \oplus \mathcal{H}_N \quad (43)$$

where $\mathcal{H}$ is the Cartan subalgebra of $\mathrm{su}(2,2)$, $\mathcal{H}_N$ is the Cartan subalgebra of $\mathrm{su}(N)$.

We express the generators of $\mathrm{U}_q(\mathcal{G}^S)$ in terms of those of $\mathrm{U}_q(\mathcal{G}^{\mathbb{C}})$. For $\mathrm{U}_q(\mathrm{su}(2,2))$ we use formulae (28)–(31), and for $\mathrm{U}_q(\mathrm{su}(N))$ formulae (11). For the latter we note that $\{\pm i\alpha_{jk} |\, 5 \leqslant j \leqslant k \leqslant N+3\}$ form the root system of $\mathrm{su}(N)$. For the generator of the $u(1)$ subalgebra in $\mathcal{G}^S_{(0)}$, $\mathcal{M}^S_{(0)}$ and $\mathcal{H}^S$ we have

$$e_N = \sum_{k=1}^{4} kH_k + \frac{4}{N} \sum_{k=5}^{N+3} (k-4-N) H_k. \quad (44)$$

Note that $e_4$ coincides with $Z_4$ described above. Next we have to express the $8N$ generators of $\mathcal{G}^S_{(1)}$. Let us denote the generators of $\tilde{\mathcal{N}}^S_{(1)} = \mathcal{G}^+_1$ by $P^\pm_{ak}$, and of $\mathcal{N}^S_{(1)} = \mathcal{G}^-_1$ by $K^\pm_{ak}$. Then we have

$$P^+_{ak} = \mathrm{i}X^+_{a,k+4} - X^+_{a+2,k+4} \qquad P^-_{ak} = X^+_{a,k+4} - \mathrm{i}X^+_{a+2,k+4}$$
$$a = 1,2, k = 1,\ldots,N \tag{45a}$$

$$K^+_{ak} = \mathrm{i}X^-_{a+2,k+4} - X^-_{a,k+4} \qquad K^-_{ak} = X^-_{a+2,k+4} - \mathrm{i}X^-_{a,k+4}$$
$$a = 1,2, k = 1,\ldots,N. \tag{45b}$$

The commutation and Hopf algebra relations of $\mathrm{U}_q(\mathrm{su}(2,2/N))$ can be explicitly written now using formulae (28)–(31), (11), (44), (45), (41), (4*a*), (38), (7). These formulae are omitted here because of the lack of space.

## Acknowledgments

The author would like to thank Professor H Reeh for hospitality at the Institute for Theoretical Physics, Göttingen University and Professor Abdus Salam for hospitality at the International Centre for Theoretical Physics, Trieste. This work was supported by the A von Humboldt Foundation, the International Atomic Energy Agency and UNESCO, and also partially by the Bulgarian National Foundation for Science, Grant $\Phi - 11$.

*Note added in proof.* (1). Recently, in [22] were obtained 16 real forms of $\mathrm{U}_q(\mathrm{so}(5,\mathbb{C}))$ were obtained by studying its automorphisms and anti-automorphisms. (2). The Lorentz algebra proposed in the present paper as an illustration of our procedure was first found in [23] as the quantum group of Liouville theory in the strong coupling regime. (3). Other deformations of $\mathrm{SO}(3,1)$ were treated in [24, 25] which appeared after the present paper was submitted.

## References

[1] Drinfeld V G 1985 *Sov. Math. Dokl.* **32** 2548; 1986 *Proc. ICM (Berkeley 1986)* vol I (New York: Academic) pp 798–820
[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63–9; 1986 *Lett. Math. Phys.* **11** 247–52
[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1987 *Preprint* LOMI Leningrad E-14-87; 1990 *Leningrad Math. J.* **1** 193–225
[4] Manin Yu I 1987 *Ann. Inst. Fourier* **37** 191–205; 1988 *Preprint* Montreal University CRM-1561
[5] Woronowicz S L 1987 *Comm. Math. Phys.* **111** 613; 1991 *Lett. Math. Phys.* **21** 35
[6] Chaichian M, Ellinas D and Kulsih P 1990 *Phys. Rev. Lett.* **65** 980
Curtright T and Zachos C 1990 *Phys. Lett.* **243B** 237
Kulish P P and Damashinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
[7] Chakrabarti A 1990 *Preprint* Ecole Polytechnique Palaiseau CPT A952.0290
Vaksman L L and Korogodsky L I 1991 *Funkt. Anal. Prilozh.* **25** 60–2
[8] Podles P and Woronowicz S L 1990 *Commun. Math. Phys.* **130** 381–431
Carow-Watamura U, Schlieker M, Scholl M and Watamura S 1990 *Z. Phys.* C **48** 159–66
[9] Schmidke W B, Wess J and Zumino B 1991 *Preprint* MPI-PAE/PTh 15/91
[10] Bourbaki N 1968 *Groupes et algèbres de Lie* ch 4–6 (Paris: Hermann)
[11] Kac V G 1977 *Adv. Math.* **26** 8–96

[12] Tolstoy V N 1990 *Proc. Quantum Groups Workshop (Clausthal, 1989) (Lecture Notes in Physics 370)* ed H D Doebner and J D Hennig (Berlin: Springer) p 118
[13] Abe E 1980 *Hopf Agebras* (Cambridge: Cambridge University Press)
[14] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 *Phys. Lett.* **264B** 331–8
[15] Dobrev V K 1989 *ICTP Trieste Internal Report* IC/89/142; 1991 *Proc. Int. Group Theory Conf. (St Andrews, 1989) (London Math. Soc. Lecture Note Series 159)* pp 87–104
[16] Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72–80
[17] Scheunert M 1992 *Lett. Math. Phys.* **24** 173–81
[18] Kac V G 1991 private communication (May)
[19] Leites D A and Serganova V 1991 private communication (May)
[20] Floreanini R, Leites D A and Vinet L 1991 *Lett. Math. Phys.* **23** 127–31
[21] Tolstoy V N 1991 private communication (July)
Khoroshkin S M and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599–617
[22] Lukierski J, Nowicki A and Ruegg H 1991 *Preprint* Geneva UGVA-DPT 91-08-740
[23] Gómez C and Sierra G 1991 *Phys. Lett.* **255B** 51–60
[24] Chakrabarti A 1992 Talk presented at XIX ICGTMP, Salamanca (28 Jun–4 Jul) unpublished
[25] Drabant B, Schlieker M, Weich W and Zumino B 1992 *Commun. Math. Phys.* **147** 625